

Non-distillable entanglement guarantees distillable entanglement

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The monogamy of entanglement is one of the basic quantum mechanical features, which says that when two partners Alice and Bob are more entangled then either of them has to be less entangled with the third party. Here we qualitatively present the converse monogamy of entanglement: given a tripartite pure system and when Alice and Bob are entangled and non-distillable, then either of them is distillable with the third party. Our result leads to the classification of tripartite pure states based on bipartite reduced density operators, which is a novel and effective way to this long-standing problem compared to the means by stochastic local operations and classical communications. Furthermore we systematically indicate the structure of the classified states and generate them. We also extend our results to multipartite states.

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I. INTRODUCTION

The monogamy of entanglement is a purely quantum phenomenon in physics [1] and has been used in various applications, such as bell inequalities [2] and quantum security [3]. In general, it indicates that the more entangled the composite system of two partners Alice (A) and Bob (B) is, the less entanglement between A (B) and the environment E there is. The security of many quantum secret protocols can be guaranteed quantitatively [3, 4]. However the converse statement generally doesn't hold, namely when A and B are less entangled, we cannot decide whether A (B) and E are more entangled. In fact even when the formers are classically correlated namely separable [5], the latters may be also separable. For example, this is realizable by the tripartite Greenberger-Horne-Zeilinger (GHZ) state.

Nevertheless, it is still important to *qualitatively* characterize the above converse statement in the light of the hierarchy of entanglement of bipartite systems. Such a characterization defines a converse monogamy of entanglement, and there is no classical counterpart. Besides, it is also expected to be helpful for treating a quantum multi-party protocol when the third party helps the remaining two parties, for it guarantees the property of one reduced density operator from another. To justify the hierarchy of entanglement, we recall six well-known conditions, i.e., the separability, positive-partial-transpose (PPT) [6, 7], non-distillability of entanglement under local operations and classical communications (LOCC) [7], reduction property (states satisfying reduction criterion) [8], majorization property [9] and negativity of condi-

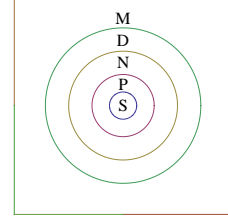


FIG. 1: Hierarchy of bipartite states in terms of five sets S , P , N , D , M . Intuitively, the sets S and P form all PPT states, the sets S , P , N , D and M form all states satisfying and violating the reduction criterion, respectively. So the five sets constitute the set of bipartite states and there is no intersection between any two sets. The strength of entanglement of the five sets becomes weak in turn, $S \subseteq P \subseteq N \subseteq D \subseteq M$.

tional entropy [8]. These conditions form a hierarchy since a bipartite state satisfying the former condition will satisfy the latter too. Therefore, the strength of entanglement in the states satisfying the conditions in turn becomes gradually *weak*.

For example, the hierarchy is closely related to the distillability of entanglement [7]. While PPT entangled states cannot be distilled to Bell states for implementing quantum information tasks, Horodecki's protocol can distill a state that violates reduction criterion [8]. That is, the former entangled state is useless as a resource while the latter entangled state is useful. So the usefulness of entangled states can be characterized by this hierarchy. Recently, a hierarchy of entanglement has been developed based on these criteria [10].

In this paper, for simplicity we consider four most important conditions, namely the separability, PPT, non-distillability and the reduction criteria. Then we establish a further hierarchy of entanglement consisting of five sets: separable states (S), non-separable PPT states (P),

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non-PPT non-distillable states (N), distillable reduction states (D), and non-reduction states (M), see Figure 1. In particular the states belonging to M are always distillable [8].

We show that when the entangled state between A and B , i.e., ρ_{AB} belongs to the set D , then the state between A (B) and E , i.e., $\rho_{AE}(\rho_{BE})$ belongs to the set D or M . Likewise when ρ_{AB} belongs to P or N , then $\rho_{AE}(\rho_{BE})$ must belong to M . Hence we can qualitatively characterize the converse monogamy of entanglement as follows: when the state ρ_{AB} is weakly entangled, then ρ_{AE} is generally strongly entangled in terms of the five sets S, P, N, D, M . These assertions follow from a corollary of Theorem 2 to be proved later.

Theorem 1 *Suppose a tripartite pure state has a non-distillable bipartite reduced state. Then another bipartite reduced state is separable if and only if it satisfies the reduction criterion.*

From this theorem, we will solve two conjectures on the existence of specified tripartite state proposed by Thapliyal in 1999 [11]. On the other hand, the theorem also helps develop the classification of tripartite pure states based on the three reduced states, each of which could be in one of the five cases S, P, N, D, M . So there are at most $5^3 = 125$ different kinds of tripartite states. Evidently, some of them do not exist due to Theorem 1. It manifests that the quantum behavior of a global system is strongly restricted by local systems. By generalizing to many-body systems, we can realize the quantum nature on macroscopic size in terms of the microscopic physical systems. This is helpful to the development of matter and material physics [12]. Hence, in theory it becomes important to totally identify different tripartite states.

To explore the problem, we describe the properties for reduced states ρ_{AB} , ρ_{BC} , and ρ_{CA} of the state $|\Psi\rangle$ by X_{AB}, X_{BC}, X_{CA} that take values in S, P, N, D, M . The subset of such states $|\Psi\rangle$ is denoted by $\mathcal{S}_{X_{AB}X_{BC}X_{CA}}$, and the subset is non-empty when there exists a tripartite state in it. For example, the GHZ state belongs to the subset \mathcal{S}_{SSS} . Furthermore as is later shown in Table I, $|\Psi\rangle$ belongs to the subset \mathcal{S}_{SSM} when the reduced state ρ_{CA} is an entangled maximally correlated state [13]. By Theorem 1, one can readily see that any non-empty subset is limited in nine essential subsets,

$$\mathcal{S}_{SSS}, \mathcal{S}_{SSM}, \mathcal{S}_{SMM}, \mathcal{S}_{PMM}, \mathcal{S}_{NMM},$$

and

$$\mathcal{S}_{DDD}, \mathcal{S}_{DDM}, \mathcal{S}_{DMM}, \mathcal{S}_{MMM}$$

. Hence up to permutation, the number of non-empty subsets for tripartite pure states is at most $21 = 1 \times 3 + 3 \times 6$. In particular, it is a long-standing open problem that whether \mathcal{S}_{NMM} exists [17]. Except the subsets generated from \mathcal{S}_{NMM} , we will demonstrate that the rest 18 subsets are indeed non-empty by explicit examples. These

subsets are not preserved under the conventional classification by the invertible stochastic LOCC (SLOCC) [14–16]. We will explain these results in Sec. III.

Furthermore, we show that the subsets form a commutative monoid and it is a basic algebraic concept. We systematically characterize the relation of the subsets by generating them under the rule of monoid in the late part of Sec. III.

We also extend our results to multipartite scenario. In particular, we introduce the multipartite separable, PPT and non-distillable states. They become pairwise equivalent when they are compatible to a pure state, see Theorem 15, Sec. IV. Finally we conclude in Sec. V.

II. UNIFICATION OF ENTANGLEMENT CRITERION

In quantum information, the following six criteria are extensively useful for studying bipartite states ρ_{AB} in the space $\mathcal{H}_A \otimes \mathcal{H}_B$.

- (1) Separability: ρ_{AB} is the convex sum of product states [5].
- (2) PPT condition: the partial transpose of ρ_{AB} is semidefinite positive [6].
- (3) Non-distillability: no pure entanglement can be asymptotically extracted from ρ_{AB} under LOCC, no matter how many copies are available [7].
- (4) Reduction criterion: $\rho_A \otimes I_B \geq \rho_{AB}$ and $I_A \otimes \rho_B \geq \rho_{AB}$ [8].
- (5) Majorization criterion: $\rho_A \succ \rho_{AB}$ and $\rho_B \succ \rho_{AB}$ [9].
- (6) Conditional entropy criterion: $H_\rho(B|A) = H(\rho_{AB}) - H(\rho_A) \geq 0$ and $H_\rho(A|B) = H(\rho_{AB}) - H(\rho_B) \geq 0$, where H is the von Neumann entropy.

It is well-known that the relation (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) holds for any state ρ_{AB} [7–9]. In particular apart from (2) \Rightarrow (3) whose converse is a famous open problem [17], all other relations are strict. We will show that these conditions become equal when we further require ρ_{BC} is non-distillable. First, under this restriction the conditions (5) and (6) are respectively simplified into (5') $\rho_A \sim \rho_{AB}$, where \sim denotes that ρ_A and ρ_{AB} have identical eigenvalues, and (6') $H(\rho_A) = H(\rho_{AB})$. Second, when ρ_{BC} is non-distillable, since $\rho_{AB} \succ \rho_A$ holds, the above two conditions (5') and (6') are equivalent. Now we have

Theorem 2 *For a tripartite state $|\Psi\rangle_{ABC}$ with a non-distillable reduced state ρ_{BC} namely condition (3), then conditions (1)-(6), (5'), and (6') are equivalent for ρ_{AB} .*

The proof is given in the appendix of this paper. We can readily get Theorem 1 from Theorem 2, and provide its operational meaning as the main result of the work.

Theorem 3 (Converse monogamy of entanglement).

Consider a tripartite state $|\Psi\rangle_{ABE}$ with entangled reduced states ρ_{AB}, ρ_{AE} and ρ_{BE} . When ρ_{AB} is a weakly entangled state P or N (D), the states ρ_{AE} and ρ_{BE} are strongly entangled states M (D or M).

To our knowledge, the converse monogamy of entanglement is another basic feature of quantum mechanics and there is no classical counterpart since classical correlation can only be "quantified". In contrast, quantum entanglement has qualitatively different levels of strength and they have essentially different usefulness from each other. For example the states in the subset N cannot be distilled while those in M are known to be distillable [8]. So only the latter can directly serve as an available resource for quantum information processing and it implies the following paradox. **A useless entangled state between A and B strengthens the usefulness of entanglement resource between A (B) and the environment.** Therefore, the converse monogamy of entanglement indicates a dual property to the monogamy of entanglement: Not only the amount of entanglement, the usefulness of entanglement in a composite system is also restricted by each other.

Apart from bringing about the converse monogamy of entanglement, Theorem 2 also promotes the study over a few important problems. For instance, the equivalence of (1) and (2) is a necessary and sufficient condition of deciding separable states, beyond that for states of rank not exceeding 4 [6, 18, 19]. Besides, the equivalence of (2) and (3) indicates another kind of non-PPT entanglement activation by PPT entanglement [20]. For later convenience, we explicitly work out the expressions of states satisfying the assumptions in Theorem 2.

Lemma 4 *The tripartite pure state with two non-distillable reduced states ρ_{AB} and ρ_{AC} , if and only if it has the form $\sum_i \sqrt{p_i} |b_i, i, i\rangle$ up to local unitary operators. In other words, the reduced state ρ_{BC} is maximally correlated.*

For the proof see Lemma 11 in [19]. We apply our results to handle two open problems in FIG. 4 of [11], i.e., whether there exist tripartite states with two PPT bound entangled reduced states, and tripartite states with two separable and one bound entangled reduced states. Here we give negative answers to these open problems in terms of Theorem 2 and Lemma 4. As the first problem is trivial, we account for the second conjecture. Because the required states have the form $\sum_{i=1}^d \sqrt{p_i} |ii\rangle |c_i\rangle$, where ρ_{AC} and ρ_{BC} are separable. So the reduced state ρ_{AB} is a maximally correlated state, which is either separable or distillable. It readily denies the second problem.

It is noticeable that the converse of Theorem 2 doesn't hold. That is, for a tripartite state $|\Psi\rangle_{ABC}$ for which conditions (1)-(6), (5'), and (6') are equivalent for ρ_{AB} , the reduced state ρ_{BC} is not necessarily non-distillable. An example is the state $|000\rangle + (|0\rangle + |1\rangle)|11\rangle$.

Finally we extend Theorem 2 for the tripartite state containing a qubit reduced state. For this purpose we introduce the following known result [21, Remark 1].

Lemma 5 *A $2 \times N$ state is PPT if and only if it satisfies the reduction criterion.*

Based on it we have

Theorem 6 *Suppose $|\Psi\rangle_{ABC}$ contains a qubit reduced state. If ρ_{BC} satisfies condition (4), then conditions (1)-(6), (5'), and (6') are equivalent for ρ_{AB} .*

Proof. When ρ_{BC} contains a qubit reduced state, the claim follows from Lemma 5 and Theorem 2. So it suffices to consider the case $\text{rank } \rho_A = 2$. Since ρ_{BC} satisfies condition (4), we obtain $\text{rank } \rho_A \geq \text{rank } \rho_B, \text{rank } \rho_C$. In other word ρ_{BC} is a two-qubit state and the claim again follows from Lemma 5 and Theorem 2. This completes the proof. \square

For tripartite states with higher dimensions, Theorem 6 does not apply anymore and we will see available examples in next section. As the concluding remark of this section, we propose the following conjecture.

Conjecture 7 *For a tripartite $3 \times 3 \times 3$ state $|\Psi\rangle_{ABC}$, suppose ρ_{BC} satisfies (4) and ρ_{AB} satisfies (5). Then ρ_{AB} also satisfies (4).*

III. CLASSIFICATION WITH REDUCED STATES

Theorem 3 says that the quantum correlation between two parties of a tripartite system is dependent on the third party. From Theorem 2 and the discussion to conjectures in [11], we can see that the tripartite pure state with some specified bipartite reduced states may not exist. This statement leads to a classification of tripartite states in terms of the three reduced states [22]. As a result, we obtain the different subsets of tripartite states in Table I in terms of tensor rank and local ranks of each one-party reduced state. In the language of quantum information, the tensor rank of a multipartite state, also known as the Schmidt measure of entanglement [23], is equal to the least number of product states to expand this state. For instance, any multiqubit GHZ state has tensor rank two. So tensor rank is bigger or equal to any local rank of a multipartite pure state. As tensor rank is invariant under invertible SLOCC [15], it has been widely applied to classify SLOCC-equivalent multipartite states recently [16].

Here we will see that, tensor rank is also essential to the classification in Table I. We will justify the statement for each subset in Table I, then we show their nonemptiness by proposing specific examples.

First the statement for the subsets $\mathcal{S}_{SSS}, \mathcal{S}_{SSM}$ and \mathcal{S}_{SMM} follows from Lemma 4, and Lemma 2 in [24], respectively. The nonemptiness readily follows from the

states $|\psi\rangle_{ABC}$ in Table I. To verify the statement for \mathcal{S}_{PMM} and \mathcal{S}_{NMM} , we propose the following result

Lemma 8 *A $M \times N$ state with rank N is non-distillable if and only if it is separable and is the convex sum of just N product states, i.e., $\sum_{i=1}^N p_i |a_i, b_i\rangle\langle a_i, b_i|$.*

Proof. It suffices to show the necessity. Let ρ be a $M \times N$ state with rank N and suppose it is non-distillable. From Lemma 11 in [19] we obtain ρ is PPT. The claim then follows from [18]. This completes the proof. \square

As a result, the PPT entangled state ρ_{AB} satisfies $d_A, d_B < \text{rank } \rho_{AB}$. So the purification of PPT entangled states ρ_{AB} is subject to the statement for subset \mathcal{S}_{PMM} in Table I. It also shows the nonemptiness of \mathcal{S}_{PMM} . A similar argument can be used to justify the statement for \mathcal{S}_{NMM} in Table I, if it really exists.

Next we study the subset \mathcal{S}_{DDD} . Since the reduced state ρ_{AB} satisfies the reduction criterion, we have $d_C \geq d_A, d_B$. By applying the same argument to other reduced states we obtain $d_A = d_B = d_C$. Since ρ_{AB} is distillable, the rest statement $r > d_A$ in Table I follows from the following observation.

Lemma 9 *Assume that $\text{rk}(\Psi) = \max\{d_A, d_B\}$. Then the conditions (1)-(4) are equivalent for ρ_{AB} .*

Proof. It suffices to show that the state ρ_{AB} is separable when it satisfies the reduction criterion. Let $|\Psi\rangle = \sum_{i=1}^{\text{rk}(\Psi)} \sqrt{p_i} |a_i, b_i, c_i\rangle$, and V_A an invertible matrix such that $V_A |i\rangle = |a_i\rangle$. Now, we focus on the pure state $|\Psi'\rangle = K V_A^{-1} \otimes I_B \otimes I_C |\Psi\rangle$, where K is the normalized constant. Then the reduced state ρ'_{AB} satisfies $\rho'_A \otimes I_B \geq \rho'_{AB}$, and hence $\rho'_A \succ \rho'_{AB}$ [9]. Since ρ'_{BC} is separable, we have $\rho'_A \sim \rho'_{AB}$. So the state ρ'_{AB} , and equally ρ_{AB} is separable in terms of Theorem 2. \square

Example 10 *It's noticeable that under the same assumption in Lemma 9, the equivalence between conditions (1)-(5) does not hold. As a counterexample, we consider the symmetric state $|\Psi\rangle = \frac{1}{\sqrt{r+1}} (\sum_{i=2}^r |iii\rangle + (|1\rangle + |2\rangle)(|1\rangle + |2\rangle)(|1\rangle + |2\rangle))$. It is symmetric and thus satisfies condition (5). On the other hand one can directly show that any reduced state of $|\Psi\rangle$ violate the reduction criterion, so it does not satisfy conditions (1)-(4). Hence $|\Psi\rangle$ belongs to the subset \mathcal{S}_{MMM} . It indicates that tensor rank alone is not enough to characterize the hierarchy of bipartite entanglement.*

Example 11 *The symmetric state $|\psi_r\rangle = \frac{1}{\sqrt{2r}}(|312\rangle + |123\rangle + |231\rangle + |213\rangle + |132\rangle + |321\rangle) + \frac{1}{\sqrt{r}} \sum_{j=4}^r |jjj\rangle$*

indicates the nonemptiness of \mathcal{S}_{DDD} for $d_A \geq 3$. To see it, it suffices to show one of the reduced states, say ρ_{AB} satisfies the reduction criterion and is distillable simultaneously. The former can be directly justified. By performing the local projector $P = |1\rangle\langle 1| + |2\rangle\langle 2|$ on system A, we obtain the resulting state $P \otimes I \rho_{AB} P \otimes I = (|12\rangle + |21\rangle)(\langle 12| + \langle 21|)$ which is a Bell state. So ρ_{AB} is distillable.

However there is no state with $d_A = 2$ in \mathcal{S}_{DDD} . The reason is that a two-qubit state satisfying the reduction criterion is also PPT by Lemma 5. Hence it must be separable in terms of Peres' condition [25]. It contradicts with the statement that ρ_{AB} is distillable.

By using a similar argument to \mathcal{S}_{DDD} , one can verify the statement for \mathcal{S}_{DDM} in Table I. A concrete example will be built by the rule of monoid later. Hence \mathcal{S}_{DDM} is nonempty.

Third, we characterize the subset \mathcal{S}_{DMM} by the tensor rank of states in this subset. It follows from the definition of reduction criterion that $d_C \geq d_A, d_B$. This observation and Lemma 9 justify the statement for \mathcal{S}_{DMM} in Table I. In order to show the tightness of these inequalities, we consider the non-emptiness of the subset \mathcal{S}_{DMM} with the boundary types $r = d_C > d_A = d_B$ and $r > d_C = d_A = d_B$.

Example 12 *To justify the former type, it suffices to consider the state $|\psi_a\rangle = \frac{1}{\sqrt{6+3a^2}}(|123\rangle + |231\rangle + |312\rangle + |21\rangle(|3\rangle + a|6\rangle) + |13\rangle(|2\rangle + a|5\rangle) + |32\rangle(|1\rangle + a|4\rangle))$, $a \in \mathbb{R}$. It obviously fulfils $r = d_C > d_A = d_B$, so the reduced states ρ_{AC} and ρ_{BC} violate the reduction criterion. Next we focus on the reduced state ρ_{AB} . By performing the local projector $P = |1\rangle\langle 1| + |2\rangle\langle 2|$ on system A, we obtain the resulting state $P \otimes I \rho_{AB} P \otimes I = (|12\rangle + |21\rangle)(\langle 12| + \langle 21|) + a^2 |21\rangle\langle 21|$. This is an entangled two-qubit state and is hence distillable [26]. On the other hand to see that ρ_{AB} fulfils the reduction criterion for any a , one need notice the facts $\rho_A = \rho_B = \frac{1}{3}I$ and the eigenvalues of ρ_{AB} are not bigger than $\frac{1}{3}$.*

In addition, an example of the latter type is constructed by the rule of monoid later. Thus, we can confirm the tightness of the above inequalities of two boundary types for \mathcal{S}_{DMM} .

To conclude, we have verified the statement and existence of all essential subsets in Table I except the \mathcal{S}_{NMM} .

Comparison to SLOCC classification. We know that there are much efforts towards the classification of multipartite state by invertible SLOCC [14–16]. Hence, it

is necessary to clarify the relation between this method and the classification by reduced states in Table I. When we adopt the former way we have no clear characteriza-

TABLE I: Classification of tripartite states $|\psi\rangle_{ABC}$ in terms of the bipartite reduced states. The table contains neither the classes generated from the permutation of parties, and nor the subset \mathcal{S}_{MMM} since for which there is no fixed relation between the tensor rank $\text{rk}(\psi)$ and *local ranks* $d_A(\Psi)$, $d_B(\Psi)$, and $d_C(\Psi)$. They are simplified to r , d_A , d_B , and d_C when there is no confusion. All expressions are up to local unitaries (LU) and all sums run from 1 to r . In the subset \mathcal{S}_{SMM} , the linearly independent states $|c_i\rangle$ are the support of space \mathcal{H}_C . Apart from the subset \mathcal{S}_{NMM} , all other eight subsets are nonempty in terms of specific examples in Sec. III.

$\mathcal{S}_{X_{AB}X_{BC}X_{CA}}$	\mathcal{S}_{SSS}	\mathcal{S}_{SSM}	\mathcal{S}_{SMM}
expression of $ \psi\rangle_{ABC}$	$\sum_i \sqrt{p_i} iii\rangle$	$\sum_i \sqrt{p_i} i, b_i, i\rangle$	$\sum_i \sqrt{p_i} a_i, b_i, c_i\rangle$
tensor rank and local ranks	$r = d_A = d_B = d_C$	$r = d_A = d_C \geq d_B$	$r = d_C \geq d_A, d_B$

$\mathcal{S}_{X_{AB}X_{BC}X_{CA}}$	\mathcal{S}_{PMM}	\mathcal{S}_{NMM}	\mathcal{S}_{DDD}
expression of $ \psi\rangle_{ABC}$	$\sum_i \sqrt{p_i} a_i, b_i, c_i\rangle$	$\sum_i \sqrt{p_i} a_i, b_i, c_i\rangle$	$\sum_i \sqrt{p_i} a_i, b_i, c_i\rangle$
tensor rank and local ranks	$r \geq d_C > d_A, d_B$	$r \geq d_C > d_A, d_B$	$r > d_C = d_B = d_A$

$\mathcal{S}_{X_{AB}X_{BC}X_{CA}}$	\mathcal{S}_{DDM}	\mathcal{S}_{DMM}
expression of $ \psi\rangle_{ABC}$	$\sum_i \sqrt{p_i} a_i, b_i, c_i\rangle$	$\sum_i \sqrt{p_i} a_i, b_i, c_i\rangle$
tensor rank and local ranks	$r > d_C = d_A \geq d_B$	$r \geq d_C \geq d_A, d_B$ $r > d_A, d_B$

tion to the hierarchy of bipartite entanglement between the involved parties, i.e., the structure of reduced states becomes messy under SLOCC. Our classification resolves this drawback. Another potential advantage of our idea is that we can apply the known fruitful results of bipartite entanglement, such as the hierarchy of entanglement to further study the classification problem.

Here we explicitly exemplify that the invertible SLOCC only partially preserves the classification in Table I. We focus on the orbit $\mathcal{O}_{r=d_A=d_B=d_C} := \{|\Psi\rangle | \text{rk}(\Psi) = d_A(\Psi) = d_B(\Psi) = d_C(\Psi)\}$, which has non-empty intersection with the subsets \mathcal{S}_{SSS} , \mathcal{S}_{SSM} , and \mathcal{S}_{SMM} . Further, since the subset \mathcal{S}_{MMM} contains the state $|\Psi_a\rangle$, it also has non-empty intersection with $\mathcal{O}_{r=d_A=d_B=d_C}$. Since all state in $\mathcal{O}_{r=d_A=d_B=d_C}$ can be converted to GHZ state by invertible SLOCC, it does not preserve the classification by reduced states. However, the subsets \mathcal{S}_{DDM} and \mathcal{S}_{DDD} are not mixed with \mathcal{S}_{SSS} , \mathcal{S}_{SSM} , and \mathcal{S}_{SMM} by invertible SLOCC.

Monoid structure. To get a further understanding of Table I from the algebraic viewpoint, we define the direct sum for subsets by $\mathcal{S}_{X_{AB}X_{BC}X_{CA}} \mathcal{S}_{Y_{AB}Y_{BC}Y_{CA}} := \mathcal{S}_{\max\{X_{AB}, Y_{AB}\} \max\{X_{BC}, Y_{BC}\} \max\{X_{CA}, Y_{CA}\}}$, where $\max\{X, Y\}$ is the larger one between X and Y in the order $S \leq P \leq N \leq D \leq M$. Therefore when $|\Psi_1\rangle \in \mathcal{S}_{X_{AB}X_{BC}X_{CA}}$ and $|\Psi_2\rangle \in \mathcal{S}_{Y_{AB}Y_{BC}Y_{CA}}$, the state $|\Psi_1 \cdot \Psi_2\rangle := |\Psi_1\rangle \oplus |\Psi_2\rangle \in \mathcal{S}_{\max\{X_{AB}, Y_{AB}\} \max\{X_{BC}, Y_{BC}\} \max\{X_{CA}, Y_{CA}\}}$. This product is commutative and in the direct sum, the subset

\mathcal{S}_{SSS} is the unit element but no inverse element exists. So the family of non-empty sets $\mathcal{S}_{X_{AB}X_{BC}X_{CA}}$ with the direct sum is an abelian monoid, which is a commutative semigroup associated with the unit.

The above analysis provides a systematic method to produce the subsets in the monoid structure, except \mathcal{S}_{NMM} whose existence is an open problem so far. Generally we have $\mathcal{S}_{SMM} = \mathcal{S}_{SSM} \mathcal{S}_{SMS}$, $\mathcal{S}_{DDM} = \mathcal{S}_{DDD} \mathcal{S}_{SSM}$, $\mathcal{S}_{DMM} = \mathcal{S}_{DDD} \mathcal{S}_{SMM}$, and $\mathcal{S}_{MMM} = \mathcal{S}_{PMM} \mathcal{S}_{MSS}$. So it is sufficient to check the non-emptiness of subsets \mathcal{S}_{SSS} , \mathcal{S}_{SSM} , \mathcal{S}_{PMM} , \mathcal{S}_{DDD} . This fact has been verified in Sec. III, and we can use the method to produce states in \mathcal{S}_{DMM} . The following is an example.

If we choose $|\Psi_1\rangle \in \mathcal{S}_{SMM}$ and $|\Psi_2\rangle \in \mathcal{S}_{DDD}$ and both have $d_A = d_B = d_C$. Then the state $|\Psi_1 \cdot \Psi_2\rangle$ verifies the non-emptiness of the subset \mathcal{S}_{DMM} with the condition $r > d_C = d_A = d_B$. The arguments have verified the existence of a boundary type mentioned in the second paragraph below Lemma 9.

IV. GENERALIZATION TO MULTIPARTITE SYSTEM

We begin by introducing the following results from [11, 27]. By fully separable states ρ of N -partite systems, we mean $\rho = \sum_i p_i \rho_{1,i} \otimes \cdots \otimes \rho_{N,i}$.

Lemma 13 *The $M \times N$ states of rank less than M or N are distillable, and consequently they are NPT.*

Lemma 14 *The N -partite state $|\psi\rangle$ has N fully separable $(N-1)$ -partite reduced states if and only if $|\psi\rangle$ is a generalized GHZ state $\sum_i \sqrt{p_i} |ii \dots i\rangle$ up to LU.*

Next, we generalize Lemma 4 and 14. For this purpose we define the n -partite non-distillable state $\rho_{1\dots n}$, in the sense that one cannot distill any pure entangled state by collective LOCC over any bipartition of parties $1, \dots, m : (m+1), \dots, n$. By "collective LOCC" we regard parties $1, \dots, m$ and $(m+1), \dots, n$ as two local parties, respectively. Similarly, we define the n -partite PPT state in the sense that any bipartition of the state is PPT. Hence, such states contain a more restrictive quantum correlation than the general multipartite state. Evidently, the multipartite PPT state is a special multipartite non-distillable state. The converse is unknown even for the bipartite case [17]. In what follows we will show the equivalence of multipartite non-distillable, PPT and fully separable states which are reduced states of a multipartite pure state.

For convenience we denote $\rho_{\bar{i}}$ as a $(N-1)$ -partite state by tracing out the party A_i from the N -partite state $|\psi\rangle$, i.e., $\rho_{\bar{i}} = \text{Tr}_i |\psi\rangle\langle\psi|$. With these definitions we have

Theorem 15 *The following four statements are equivalent for a N -partite state $|\psi\rangle$.*

- (1) $|\psi\rangle$ has n non-distillable $(N-1)$ -partite reduced states $\rho_{\bar{1}}, \dots, \rho_{\bar{n}}$, $N \geq n \geq 2$;
- (2) $|\psi\rangle$ has n PPT $(N-1)$ -partite reduced states $\rho_{\bar{1}}, \dots, \rho_{\bar{n}}$, $N \geq n \geq 2$;
- (3) $|\psi\rangle$ has n fully separable $(N-1)$ -partite reduced states $\rho_{\bar{1}}, \dots, \rho_{\bar{n}}$, $N \geq n \geq 2$;
- (4) $|\psi\rangle = \sum_i \sqrt{p_i} |i\rangle^{\otimes n} |b_{i,n+1}\rangle \otimes \dots \otimes |b_{i,N}\rangle$ up to LU.

Proof. Suppose $|\psi\rangle$ is of dimensions $d_1 \times \dots \times d_N$. The direction (4) \rightarrow (3) \rightarrow (2) \rightarrow (1) is evident. To show (1) \rightarrow (4), suppose $\rho_{\bar{1}}, \dots, \rho_{\bar{n}}$ are non-distillable. By using Lemma 13 and the definition of multipartite non-distillable states, we can obtain $d := d_1 = \dots = d_n \geq d_{n+1}, \dots, d_N$. By combining the parties A_3, \dots, A_N into one party, we obtain a tripartite state satisfying Lemma 4. Hence we have $|\psi\rangle = \sum_{i=1}^d \sqrt{p_i} |ii\rangle |\varphi_i\rangle$ where $|\varphi_i\rangle \in \bigotimes_{i=3}^N \mathcal{H}_i$.

We show that $|\varphi_i\rangle$ are fully product states. The proof is by contradiction. Suppose there is, say $|\varphi_1\rangle$ which is not fully factorized. So we can write it as a bipartite entangled state in the space $\mathcal{H}_C \otimes \mathcal{H}_D$. In other word we have the 4-partite state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ and the tripartite reduced state $\sigma_{2,C,D} = \sum_{i=1}^d |i\rangle\langle i|_2 \otimes |\varphi_i\rangle\langle\varphi_i|_{CD}$. By performing the projector $|1\rangle\langle 1|$ on space \mathcal{H}_2 , we can distill a pure entangled state $|\varphi_1\rangle$ from $\sigma_{2,C,D}$. It contradicts with the assumption on $\rho_{\bar{1}}, \dots, \rho_{\bar{n}}$. So every state $|\varphi_i\rangle$ has to be fully factorized and up to LU,

we have

$$|\psi\rangle = \sum_{i=1}^d \sqrt{p_i} |ii\rangle \bigotimes_{j=3}^N |a_{i,j}\rangle. \quad (1)$$

Next, we combine parties A_1, A_4, \dots, A_N together and make $|\psi\rangle$ a new tripartite state. Because $\rho_{\bar{3}}$ is non-distillable and any entangled maximally correlated state is distillable [13], the states $|a_{i,3}\rangle$ have to be orthonormal. In a similar way we can prove the states $|a_{i,j}\rangle$, $j = 4, \dots, n$ are orthonormal, respectively. So we have justified the statement. This completes the proof of (1) \rightarrow (4). So all four statements (1),(2),(3),(4) are equivalent. \square

The following result is a stronger version of Lemma 14.

Corollary 16 *The N -partite state $|\psi\rangle$ has N non-distillable $(N-1)$ -partite reduced states if and only if it is a generalized GHZ state $|\psi\rangle = \sum_i \sqrt{p_i} |i\rangle^{\otimes N}$ up to LU.*

V. CONCLUSIONS

We have proposed the converse monogamy of entanglement such that when Alice and Bob are weakly entangled, then either of them is generally strongly entangled with the third party. We believe that the converse monogamy of entanglement is an essential quantum mechanical feature and it promises a wide application in deciding separability, entanglement distillation and quantum cryptography. Our result presents two main open questions: First, can we propose a concrete quantum scheme by the converse monogamy of entanglement? Such a scheme will indicate a new essential difference between the classical and quantum rules, just like that from quantum cloning [28] and the negative conditional entropy [29]. Second, different from the monogamy of entanglement which relies on the specific entanglement measures [3], the converse monogamy of entanglement only relies on the strength of entanglement. So can we get a better understanding by adding other criteria on the strength of entanglement such as the non-distillability?

We also have shown tripartite pure states can be sorted into 21 subsets and they form an abelian monoid. It exhibits a more canonical and clear algebraic structure of tripartite system compared to the conventional SLOCC classification [14]. More efforts from both physics and mathematics are required to understand such structure.

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Appendix

We prove Theorem 2 based on the following preliminary lemma.

Lemma 17 *Consider a tripartite state $|\Psi_{ABC}\rangle$ with a separable reduced state ρ_{BC} . When ρ_{AB} satisfies the condition (6'), it also satisfies the condition (1).*

Proof. Due to separability, ρ_{BC} can be written by $\rho_{BC} = \sum_i p_i |\phi_i^B, \phi_i^C\rangle\langle\phi_i^B, \phi_i^C|$. We introduce the new system \mathcal{H}_D with the orthogonal basis e_i^D and the tripartite extension $\rho_{BCD} := \sum_i p_i |\phi_i^B, \phi_i^C, e_i^D\rangle\langle\phi_i^B, \phi_i^C, e_i^D|$. The monotonicity of the relative entropy $D(\rho||\sigma) := \text{Tr}(\rho \log \rho - \rho \log \sigma)$ implies that

$$\begin{aligned} 0 &= H(\rho_C) - H(\rho_{BC}) = D(\rho_{BC}||I_B \otimes \rho_C) \\ &\leq D(\rho_{BCD}||I_B \otimes \rho_{CD}) = \sum_i p_i (\log p_i - \log p_i) = 0, \end{aligned}$$

where the first equality is from Condition (6'). So the equality holds in the above inequality. According to Petz's condition [30], the channel $\Lambda_C : \mathcal{H}_C \mapsto \mathcal{H}_C \otimes \mathcal{H}_D$ with the form $\Lambda_C(\sigma) := \rho_{CD}^{1/2} ((\rho_C^{-1/2} \sigma \rho_C^{-1/2}) \otimes I_D) \rho_{CD}^{1/2}$ satisfies $\text{id}_B \otimes \Lambda_C(\rho_{BC}) = \rho_{BCD}$. We introduce the system \mathcal{H}_E as the environment system of Λ_C and the isometry $U : \mathcal{H}_C \mapsto \mathcal{H}_C \otimes \mathcal{H}_D \otimes \mathcal{H}_E$ as the Stinespring extension of Λ_C . So the state $|\Phi_{ABCDE}\rangle := I_{AB} \otimes U |\Psi_{ABC}\rangle$ satisfies $\rho_{BCD} = \text{Tr}_{AE} |\Phi_{ABCDE}\rangle\langle\Phi_{ABCDE}|$. By using an orthogonal basis $\{e_i^{AE}\}$ on $\mathcal{H}_A \otimes \mathcal{H}_E$ we can write up the state $|\Phi_{ABCDE}\rangle = \sum_i \sqrt{p_i} |\phi_i^B, \phi_i^C, e_i^D, e_i^{AE}\rangle$. Then, the state $\rho_{AB} = \text{Tr}_{CDE} |\Phi_{ABCDE}\rangle\langle\Phi_{ABCDE}| = \sum_i p_i |\phi_i^B\rangle\langle\phi_i^B| \otimes \text{Tr}_E |e_i^{AE}\rangle\langle e_i^{AE}|$ is separable. This completes the proof. \square

Due to Lemma 17 and the equivalence of conditions (5') and (6'), it suffices to show that when ρ_{BC} is non-distillable and ρ_{AB} satisfies (5'), ρ_{BC} is separable. From (5') for ρ_{AB} , it holds that $\text{rank } \rho_{BC} = d_A = \text{rank } \rho_{AB} = d_C$. It follows from [19, Theorem 10] that ρ_{BC} has to be PPT. So ρ_{BC} is separable by Lemma 8, and we have Theorem 2.

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